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**THE HOSE INSTABILITY OF A FINITE, PINCHED, RELATIVISTIC, ELECTRON BEAM
PENETRATING AN INFINITE, FIELD-FREE, TEMPERATE PLASMA**

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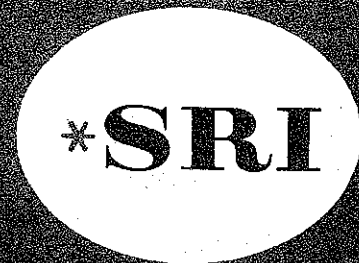
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By: Howard Chang

S T A N F O R D R E S E A R C H I N S T I T U T E

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SUMMARY

The hose instability of a uniform, cylindrical electron beam pinched by its self-magnetic field and penetrating an infinite, field-free, temperate plasma is analyzed. The tenuous electron beam is infinitely long and is streaming through the stationary, dense, uniform plasma at a nearly constant, relativistic velocity. The beam is assumed to be an incompressible fluid, whose behaviour is adequately described by the equations of magnetohydrodynamics. The electromagnetic influence of the plasma is accounted for crudely by a complex, scalar conductivity, whose phase is crucial in determining the stability of the beam-plasma system when subjected to infinitesimal perturbations of the form $q_1(r, \theta, z, t) = \tilde{q}_1(r) \exp i[\omega t - kz - \theta]$. For simplicity the discussion is confined to the long-wavelength region, when $kr_0 \ll 1$, and to the low-frequency region, when $\omega \ll kv_0$, $|\omega - kv_0| \ll \omega_B$ and $\left| \frac{\omega^2}{\omega_p^2 + \omega_{pb}^2} \right| \ll 1$. By using the jump-conditions of magnetohydrodynamics, Maxwell's equations, and a normal mode analysis, the dispersion equation, $(\omega - kv_0)^2 = \omega_B^2 [1 - i\pi J_1(ik_0 r_0) H_1^1(ik_0 r_0)]$, Eq. (60), is obtained and its solution discussed. Strange to say, the identical dispersion equation is obtained using other entirely different but reasonable models of the beam-plasma system, of greater sophistication than the simple, macroscopic model used here [7, 8, 9].

I INTRODUCTION

Consider a uniform, cylindrical electron beam which is penetrating a stationary, field-free, infinite, temperate* plasma at a nearly constant relativistic velocity. The plasma's charge density is assumed to be so much higher than the beam's charge density that quasi-charge neutrality is preserved at all times. The pinch magnetic field of the beam is deemed important in determining its behaviour, and in the equilibrium state it is sufficient to hold the beam together against its random, transverse kinetic energy and multiple scattering effects in the plasma. The equilibrium configuration of the beam is invariant to translations in the z direction, the direction of the beam motion, and the cylindrical beam surface, where the electrons undergo infinitesimal transverse displacements, r_1 , is defined by

$$r = r_0 + r_1(r, \theta, z, t) \quad . \quad . \quad . \quad (1)$$

If r_1 and all other perturbed quantities, $q_1(r, \theta, z, t)$, are Fourier analyzed so that

$$q_1(r, \theta, z, t) = \tilde{q}_1(r) \exp i[\omega t - kz - m\theta] \quad , \quad (2)$$

it follows from the stationary, cylindrically symmetric nature of the unperturbed system and the system of linear differential equations obtained following the usual linearization process in which the products

* Following Allis and Buchsbaum [1, p. 18], we shall define a temperate plasma as a plasma whose induced, thermal, and phase velocities satisfy the inequality

$$v_E \ll v_T \ll v_{Ph} = \omega/k \quad .$$

Except at resonances, most plasmas are temperate.

of two or more perturbation quantities are neglected, that each component characterized by the parameters m , k , ω is decoupled from the rest.

Of the numerous different kinds of instabilities which can disrupt this beam-plasma system, we will concern ourselves only with the so-called transverse instabilities. It is easily shown that the sausage or $m = 0$ mode causes harmonic variation of the beam radius with distance along the axis and with time. This azimuthally symmetric pulsation of the beam is phased in such a manner that the induced plasma currents exert forces on the beam electrons in such a direction and phase as to feed the pulsation. The hose or $m = 1$ mode represents a transverse displacement of the beam with the induced plasma currents again contributing to the instability. Higher values of m are more difficult to picture and will not concern us. They are not different in principle from the $m = 1$ mode, to which we will confine this discussion.

To analyze the stability of the beam-plasma system, we must express the perturbed electromagnetic field vectors in terms of the perturbed volume and surface currents of the beam, the plasma currents, and the steady state parameters such as the beam radius, beam velocity, and plasma frequencies. It is assumed that the electrons and ions of the fully ionized, temperate plasma obey a simple, linearized equation of motion in the perturbed electric field with a friction term depending on the relative electron-ion motion. To keep the analysis within reasonable bounds, collisions between like particles, magnetic forces, and thermal effects in the plasma are all neglected. Collisions between the electron beam and the plasma are proportional to the beam current, and for a weak beam the neglect of such collisions is justified. The tenuous, relativistic

beam is assumed to be adequately described by the linearized equations of magnetohydrodynamics (MHD). Instead of the usual adiabatic law,

$$\frac{1}{p} \frac{dp}{dt} = \frac{\gamma}{\rho} \frac{d\rho}{dt} \quad , \quad (3)$$

as the equation of state for the beam, we shall simplify the algebra greatly with the much less valid incompressible flow assumption that

$$\frac{d\rho}{dt} = \nabla \cdot \bar{v} = 0 \quad (4)$$

or $\gamma \rightarrow$ infinity. We shall rationalize this bald assumption by arguing that the incompressible flow is a possible mode of motion for a compressible fluid so that an instability found for an incompressible fluid is a possible instability for the more realistic compressible system. By integrating Maxwell's equations and the equations of MHD across the beam-plasma intersurface, which is assumed to be a sharp boundary, the so-called jump-conditions will be obtained. By requiring that these jump-conditions be satisfied, the dispersion equation, Eq. (60), valid for low frequencies and long wavelength, will emerge. A brief analysis of the dispersion equation concludes the report. Definitions of the more important symbols are given in the glossary located at the end of the report. The MKS system of units is used throughout.

II PLASMA DYNAMICS

With the assumptions stated in the Introduction, the particles of the temperate plasma consisting of electrons of mass m and ions of mass M , which are at rest in equilibrium with densities n_0 and N_0 respectively,

obey Newton's equation of motion [2, p. 750]

$$i\omega n_o m \bar{v}_e = -n_o e \bar{E} - \nu \left(\frac{n_o N_o m M}{n_o m + N_o M} \right) (\bar{v}_e - \bar{v}_i) \quad (5)$$

$$i\omega N_o M \bar{v}_i = N_o e \bar{E} + \nu \left(\frac{n_o N_o m M}{n_o m + N_o M} \right) (\bar{v}_e - \bar{v}_i), \quad (6)$$

where ν is an averaged, velocity-independent, electron-ion collision frequency. In writing these equations we have tacitly used the fact that the plasma is infinite in extent and that waves excited in it vary as $\exp i(\omega t - \bar{k} \cdot \bar{r})$. This permits the replacement of $\frac{d}{dt} \doteq \frac{\partial}{\partial t} = i\omega$.

The current in the plasma

$$\bar{J}_p = e(N_o \bar{v}_i - n_o \bar{v}_e) = \left(\frac{\alpha n_o e^2}{\mu(i\omega + \nu)} + \frac{(n_o - N_o)^2 e^2}{i\omega(n_o m + N_o M)} \right) \bar{E} = \sigma \bar{E}, \quad (7)$$

where $\mu = mM/(m + M)$ is the reduced mass. The conductivity, σ , can be redefined in terms of the plasma frequencies for the relative electron-ion motion, ω_e , and for the motion of electrons and ions together, ω_i .

Calling

$$\omega_e^2 = \frac{\alpha n_o e^2}{\epsilon_o \mu} \quad \text{and} \quad \omega_i^2 = \frac{(n_o - N_o)^2 e^2}{\epsilon_o (n_o m + N_o M)}, \quad (8)$$

$$\sigma = \frac{\epsilon_o}{i\omega} \left(\omega_e^2 \frac{\omega}{\omega - i\nu} + \omega_i^2 \right) = \frac{\epsilon_o}{i\omega} \omega_p^2. \quad (9)$$

Assuming that the beam is weak so that the quasi-neutrality of the plasma is never disturbed due to the passage of the beam through the plasma,

$$N_o - n_o \doteq N_{bo}, \quad (10)$$

and we can write

$$\omega_i^2 = \frac{N_{bo} e^2}{\epsilon_o M_{eff}} \quad (11)$$

where

$$M_{eff} = \frac{n_o m + N_o M}{N_o - n_o} \doteq \frac{N_o}{N_{bo}} M \gg M \quad (12)$$

Notice that the conductivity, σ , is a complex quantity. Its phase, Θ , is approximately

$$\Theta = - \text{arc tan } (\omega/\nu) \quad (13)$$

and is critical in determining the stability of the beam-plasma system.

In a fully ionized plasma, three types of interactions between the charged particles are present: electron-electron (ν_{ee}), electron-ion (ν_{ei}), and ion-ion (ν_{ii}) interactions. Delcroix [3, p. 113] has shown that they are related by

$$\nu = \nu_{ei} \doteq \nu_{ee} \doteq \sqrt{\frac{M}{m}} \nu_{ii} = 4\pi n_o \langle v_{th} \rangle b_o^2 \log (h/b_o) \quad (14)$$

where $\langle v_{th} \rangle =$ the thermal velocity of the plasma electrons $= \sqrt{(3KT_p/m)}$, h is the Debye length, and b_o is the averaged, critical impact parameter for a deflection of 90° , $b_o = e^2/(12\pi\epsilon_o KT_p)$. Because $m/M \doteq 1/1840 \ll 1$, we are fully justified in neglecting ν_{ii} . Equation (14) shows that ν can take on a large range of values by varying n_o and/or T_p but in such a way that we are always dealing with a non-relativistic, classical plasma. After specifying the beam equilibrium parameters, and by varying ν , it will be shown that the dispersion equation,

Eq. (60), can have complex roots with negative imaginary parts. In other words, for certain values of ν , the beam-plasma system is unstable.

III BEAM DYNAMICS

The behaviour of the uniform, cylindrical electron beam is best described in cylindrical coordinates with the positive z axis parallel to the gross beam velocity. We will assume that within the beam, where $r \leq r_0 + r_1$, the following equations are satisfied:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0, \quad (15)$$

$$\nabla \cdot \bar{v} = 0 \quad (16)$$

$$N_b \frac{d\bar{p}}{dt} = N_b \left(\frac{\partial}{\partial t} + \bar{v} \cdot \nabla \right) \bar{p} = -N_b e \bar{E} + \bar{J}_b \times \bar{B} - \nabla P \quad (17)$$

$$\text{where } \bar{J}_b = -N_b e \bar{v}, \quad \bar{p} = \gamma m \bar{v}, \quad \bar{v} = \frac{c \bar{p}}{\sqrt{(m^2 c^2 + p^2)}}. \quad (18)$$

Additionally, Maxwell's equations are valid, and for the beam-plasma system they can be simplified to:

$$\nabla \cdot \bar{E} = 0, \quad (19) \quad \nabla \cdot \bar{B} = 0, \quad (20)$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \quad (21) \quad \nabla \times \bar{B} = \mu_0 (\bar{J}_p + \bar{J}_b). \quad (22)$$

Some brief remarks about the validity of Eqs. (15)-(22) may be helpful. Equation (16) states that the electron fluid is incompressible. With the aid of the equation of continuity, Eq. (15), it can be written equivalently as

$$\frac{d\rho}{dt} = 0. \quad (23)$$

Although an incompressible fluid is only a convenient fiction of the theoretician which permits him to obtain simple results, and its existence violates well-established physical laws, we have justified the use of Eq. (16) by the remarks following Eq. (4). The use of Laplace's equation, Eq. (19), in place of Poisson's equation, is justified by the argument that

$$\rho_{b1} \doteq \frac{v_1}{v_0 + \frac{\omega}{k}} \rho_0 \doteq \frac{v_1}{c + \frac{\omega}{k}} \rho_0 \doteq 0, \quad \rho_{p1} \doteq \frac{v_1}{\frac{\omega}{k}} \rho_0 \doteq 0, \quad (24)$$

for $v_1/c \lll 1$, and $v_1/(\omega/k) \ll 1$ in a temperate plasma. In Ampere's Law, Eq. (21), we have neglected the displacement current relative to the conduction current, for

$$\left| \frac{J_{\text{dis.}}}{J_{\text{cond.}}} \right| = \left| \frac{\epsilon_0 \omega E}{(\sigma_p + \sigma_b) E} \right| = \left| \frac{\omega^2}{(\omega_p^2 + \omega_{pb}^2)} \right| \lll 1, \quad (25)$$

at the low frequencies which interest us. (In the above, unperturbed quantities are denoted by the subscript zero and perturbed quantities by the subscript one. Thus $\bar{v} = \bar{v}_0 + \bar{v}_1$ with $v_1/v_0 \lll 1$.)

Call \bar{n} the outward directed unit normal at the beam surface. It can be shown that \bar{n} satisfies the following equation [4, p. 70-73]:

$$\frac{d\bar{n}}{dt} = \bar{n} \times [\bar{n} \times (\nabla \bar{v}) \cdot \bar{n}] \quad (26)$$

Integrating Eqs. (16), (17), and (19)-(22) across the beam-plasma intersurface, which is assumed to be a sharp boundary, the following jump-conditions are obtained:

$$\bar{n} \cdot [\bar{v}] = 0, \quad (16a)$$

$$\mathbf{J}^* \times (\bar{\mathbf{B}}^+ + \bar{\mathbf{B}}^-)/2 - \bar{\mathbf{n}}[\mathbf{P}] = 0, \quad (17a)$$

$$\bar{\mathbf{n}} \cdot [\bar{\mathbf{E}}] = 0, \quad (19a)$$

$$\bar{\mathbf{n}} \cdot [\bar{\mathbf{B}}] = 0, \quad (20a)$$

$$\bar{\mathbf{n}} \times [\bar{\mathbf{E}}] = 0, \quad (21a)$$

$$\bar{\mathbf{n}} \times [\bar{\mathbf{B}}] = \bar{\mathbf{J}}^* = \int_{r_0 - \epsilon}^{r_0 + \epsilon} (\bar{\mathbf{J}}_p + \bar{\mathbf{J}}_b) dr, \quad \epsilon > 0 \quad (22a)$$

$$\text{Lim } \epsilon \rightarrow 0$$

where

$$[\mathbf{P}] = \mathbf{P}^+ - \mathbf{P}^- = \mathbf{P}(r_0 + \epsilon) - \mathbf{P}(r_0 - \epsilon), \quad \epsilon/r_0 \lll 1, \quad (27)$$

$$[\bar{\mathbf{B}}] = \bar{\mathbf{B}}^+ - \bar{\mathbf{B}}^- = \bar{\mathbf{B}}(r_0 + \epsilon) - \bar{\mathbf{B}}(r_0 - \epsilon), \quad \epsilon/r_0 \lll 1,$$

$$[\bar{\mathbf{E}}] = \bar{\mathbf{E}}^+ - \bar{\mathbf{E}}^- = \bar{\mathbf{E}}(r_0 + \epsilon) - \bar{\mathbf{E}}(r_0 - \epsilon), \quad \epsilon/r_0 \lll 1.$$

Further progress is possible only if the above equations are linearized. If $q(r, \theta, z, t)$ is any quantity of interest, we shall make the basic assumption that $q(r, \theta, z, t) = q_0(r, \theta, z) + q_1 = q_0 + \tilde{q}_1(r) \exp i[\omega t - kz - \theta]$, Eq. (2), where q_0 is the equilibrium value of q and q_1 is the perturbed value with $|q_1/q_0| \lll 1$.

We first exploit the fact that the z velocity of the beam is approximately constant and that it is much larger than the transverse velocity, \bar{v}_t . Linearizing Eq. (17) we get,

$$N_{b_0} m \gamma_0 \frac{d}{dt} v_r = - e N_{b_0} E_r + (\bar{\mathbf{J}}_{b_0} \times \bar{\mathbf{B}})_r - \frac{\partial P}{\partial r}, \quad (28a)$$

$$N_{b_0} m \gamma_0 \frac{d}{dt} v_\theta = -e N_{b_0} E_\theta + (\bar{J}_{b_0} \times \bar{B})_\theta - \frac{1}{r} \frac{\partial P}{\partial \theta}, \quad (28b)$$

$$N_{b_0} m \gamma_0^3 \frac{d}{dt} v_z = -e N_{b_0} E_z - \frac{\partial P}{\partial z}, \quad (28c)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$

$$\gamma_0 = (1 - v_0^2/c^2)^{-\frac{1}{2}} \quad \text{and} \quad \bar{J}_{b_0} = -N_{b_0} e v_0 \bar{e}_z. \quad (29)$$

A simple application of Ampere's Law, Eq.(22), yields the equation

$$\bar{B}_0 = \frac{\mu_0 J_{b_0} r}{2} \bar{e}_\theta, \quad r \leq r_0 \quad (30)$$

$$= \frac{\mu_0 J_{b_0} r_0^2}{2r} \bar{e}_\theta, \quad r \geq r_0$$

There is no static electric field, $E_0 = 0$, and Eq. (17) requires that

$$J_{b_0} B_0 + \frac{d P_0}{dr} = 0, \quad \text{at } r = r_0.$$

Using Eqs. (29) and (30), we find that

$$P_0 = \frac{\mu_0 J_{b_0}^2}{4} (r_0^2 - r^2). \quad (31)$$

The other equilibrium values are obvious. They are

$$E_0 = J_{p_0} = J_0^* = 0, \quad \bar{n}_0 = \bar{e}_r, \quad \bar{p}_0 = \gamma_0 m v_0 \bar{e}_z. \quad (32)$$

The total beam current consists of three terms,

$$\bar{J}_b = \bar{J}_{b_0} + \bar{J}_{b_v} + \bar{J}_b^* \delta(r - r_0), \quad (33)$$

where $\bar{J}_{b_0} = -eN_{b_0} v_0 \bar{e}_z$, \bar{J}_{b_v} is the perturbed volume current, and \bar{J}_b^* is the perturbed surface current arising mainly from the slight radial movement of the electrons at the beam's surface which are bodily transported in the z direction at velocity v_0 . Consistent with our neglect of surface charge densities [the replacement of Poisson's equation by Laplace's equation, Eq. (19)] we shall assume that $\bar{J}_{b_v} = 0$. Considering the genesis of \bar{J}_b^* it is reasonable to expect that

$$\bar{J}_b^* = c_0 J_{b_0} \exp i[\omega t - kz - \theta] \bar{e}_z, \quad (34)$$

where c_0 is a constant.

Linearizing Eqs. (16), (19)-(22), and the jump conditions (16a)-(22a), we get

$$\nabla \cdot \bar{v} = 0, \quad (16a) \quad \nabla \cdot \bar{E} = 0, \quad (19a)$$

$$\nabla \cdot \bar{B} = 0, \quad (20a) \quad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \quad (21a)$$

$$\nabla \times \bar{B} = \mu_0 \left(\sigma \bar{E} + J_b^* \delta(r - r_0) \right) \quad (22a)$$

$$\bar{e}_r \cdot \bar{v} + \bar{n} \cdot \bar{v}_0 = 0, \quad (16a')$$

$$J_b^* \times \bar{B}_0 \text{ (at } r = r_0) - \bar{e}_r P^- = 0, \quad (17a')$$

$$\bar{e}_r \cdot [\bar{E}] = 0, \quad (19a') \quad \bar{e}_r \cdot [\bar{B}] = 0, \quad (20a')$$

$$\bar{e}_r \times [\bar{E}] = 0, \quad (21a') \quad \bar{e}_r \times [\bar{B}] = \bar{J}_b^*. \quad (22a')$$

In obtaining the above equations, we have used Eqs. (7) and (30)-(32), and, to avoid the troublesome practice of putting the subscript one on perturbed quantities, have omitted them entirely so that henceforth all quantities without subscripts are perturbed quantities.

IV SOLUTION OF THE WAVE EQUATION FOR \bar{E}

Our stability analysis has been resolved to the solution of the wave equation for \bar{E} ,

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) + i\mu_o \omega_o \bar{E} = -i\mu_o \omega J_b^* \delta(r - r_o), \quad (35)$$

which satisfies the boundary conditions Eq. (16a)-(22a'), and with all perturbed quantities given by Eq. (2). Let

$$\bar{E} = \bar{\psi}(r) \exp i(\omega t - kz - \theta). \quad (36)$$

Written in component form, Eq. (35) is [5, p. 116]

$$\frac{i}{r} \left[\frac{1}{r} \frac{d}{dr} (r\psi_\theta) - \frac{i}{r} \psi_r \right] + (k_o^2 + k^2) \psi_r + ik \frac{\partial}{\partial r} \psi_z = 0, \quad (37)$$

$$- \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\psi_\theta) - \frac{i}{r} \psi_r \right] + (k_o^2 + k^2) \psi_\theta - \frac{k}{r} \psi_z = 0, \quad (38)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k_o^2 \right] \psi_z - ik \left[\frac{1}{r} \frac{d}{dr} (r\psi_r) + \frac{i}{r} \psi_\theta \right] \quad (39)$$

$$= i\mu_o c_o \omega J_b^* \delta(r - r_o)$$

where

$$k_0^2 = i\mu_0 \omega \sigma = i(\text{skin-depth})^{-2} . \quad (40)$$

The solution of these coupled differential equations for $\bar{\psi}$, (37-39), is too difficult a task, so we will solve them in the long-wavelength approximation when $kr_0 \ll 1$. In the long-wavelength limit, Eqs. (37)-(39) simplify to

$$\frac{i}{r} \left[\frac{1}{r} \frac{d}{dr} (r\psi_\theta) - \frac{i}{r} \psi_r \right] + k_0^2 \psi_r = 0, \quad (37a)$$

$$-\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\psi_\theta) - \frac{i}{r} \psi_r \right] + k_0^2 \psi_\theta = 0, \quad (38a)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k_0^2 \right] \psi_z = i\mu_0 c_0 r_0 \omega \frac{J_0 \delta(r - r_0)}{r_0} . \quad (39a)$$

Notice that Eqs. (37a) and (38a) do not involve ψ_z but only ψ_r and ψ_θ , and that Eq. (39a) is an equation for ψ_z alone. The bonus of our $k \approx 0$ assumption is that we have partially decoupled these differential equations. A laborious calculation shows that the only proper solutions of Eqs. (37a) and (38a) which satisfy the boundary conditions are

$$\psi_r(r) = \psi_\theta(r) \equiv 0 . \quad (41)$$

The homogeneous solution of Eq. (39a) is the cylindrical (Bessel) function, $Z_1(ik_0 r)$ [6, p. 146]. For the particular solution, we appeal to the theory of Green's functions [5, Chapter 7, especially pages 826 ff.] and find that in the region $r \leq r_0$,

$$\psi_z(r) = \frac{\pi}{2} \omega_{b_0} c_{r_0} \omega_{b_0} H_1^1(ik_0 r) J_1(ik_0 r) \quad , \quad r \leq r_0 \quad (42)$$

where $H_1^1(z)$ and $J_1(z)$ are Hankel functions of the first kind and Bessel functions of order unity respectively.

With $\psi_z(r)$ given by Eq. (42), Eqs. (36) and (21a) can be written out explicitly as

$$\bar{E} = \psi_z \exp i[\omega t - kz - \theta] \bar{e}_z \quad , \quad (43)$$

$$\begin{aligned} \bar{B} &= \frac{i}{\omega} \nabla \times \bar{E} = \frac{i}{\omega} \left[-\frac{i}{r} \psi_z \bar{e}_r + \frac{d\psi_z}{dr} \bar{e}_\theta \right] \exp i[\omega t - kz - \theta] \\ &= \frac{1}{\omega} \left[\frac{\bar{e}_r}{r} - \left(k_0 \frac{J_0(ik_0 r)}{J_1(ik_0 r)} + \frac{i}{r} \right) \bar{e}_\theta \right] \psi_z(r) \exp i[\omega t - kz - \theta] \end{aligned} \quad (44)$$

V DERIVATION OF THE DISPERSION EQUATION

To obtain the dispersion equation, we simply insert Eqs. (43) and (44) into Eqs. (28a, b, c) and pick out the solution which satisfies the jump-conditions. In particular, satisfying Eq. (17a') will yield the dispersion equation, Eq. (60). Performing the indicated substitutions we get

$$i(\omega - kv_0) N_{b_0} m \gamma_0 v_r = \left(\frac{k_0 J_0(ik_0 r)}{J_1(ik_0 r)} + \frac{i}{r} \right) \frac{J_{b_0} \psi_z}{\omega} - \frac{\partial \tilde{P}}{\partial r} \quad , \quad (45)$$

$$i(\omega - kv_0) N_{b_0} m \gamma_0 v_\theta = \frac{J_{b_0} \psi_z}{r\omega} + \frac{i\tilde{P}}{r} \quad , \quad (46)$$

$$i(\omega - kv_0)N_{b_0}m\gamma_0^3v_z = -eN_{b_0}\psi_z + ik\tilde{P}. \quad (47)$$

To solve these equations, we copy a trick used by Mjolsness [7, p. 31] and argue that as $v_z/v_0 \lll 1$, the exact r dependence of v_z is unimportant. If we replace Eq. (47) by

$$i(\omega - kv_0)N_{b_0}m\gamma_0^3v_z = \frac{J_{b_0}k\psi_z}{\omega} + ik\tilde{P}, \quad (48)$$

then Eqs. (45), (46), and (47) are components of the vector equation

$$i(\omega - kv_0)N_{b_0}m\gamma_0^3\bar{v} = \frac{i}{\omega}J_{b_0}\nabla[\psi_z \exp i(\omega t - kz - \theta)] - \nabla[\tilde{P}(r)\exp i(\omega t - kz - \theta)]. \quad (49)$$

Equation (49) relates \bar{v} and $\tilde{P}(r)$, both unknown quantities. A little study of it suggests the ansatz

$$\bar{v} = \nabla[R(r) \exp i(\omega t - kz - \theta)], \quad (50)$$

and inserting Eq. (50) into Eq. (49), we learn that

$$R(r) = \frac{i\omega\tilde{P}(r) + J_{b_0}\psi_z(r)}{N_{b_0}\gamma_0^3m\omega(\omega - kv_0)}. \quad (51)$$

To obtain $R(r)$, we invoke the incompressibility condition, Eq. (16),

$$\nabla \cdot \bar{v} = \nabla^2[R(r) \exp i(\omega t - kz - \theta)] = 0,$$

or

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k^2 \right) R(r) = 0 \quad 0 \leq r \leq r_0 \quad (52)$$

The modified Bessel functions of the first and second kind, $I_1(kr)$ and $K_1(kr)$, satisfy Eq. (52), so

$$R(r) = 2AI_1(kr) + BK_1(kr) ,$$

where A and B are arbitrary constants. As $R(0)$ is finite, we must put $B = 0$. Recall that we are confining ourselves to the long-wavelength limit so that $kr_0 \lll 1$, and for small arguments $I_1(kr) \doteq kr/2$. Thus an approximate solution of Eq. (52) which satisfies the boundary conditions is

$$R(r) = Akr \quad kr \lll 1 \quad (53)$$

Putting Eq. (53) into Eq. (50), we get

$$\bar{v} = Ak(\bar{e}_r - i\bar{e}_\theta - ikr\bar{e}_z) \exp i(\omega t - kz - \theta) \quad (54)$$

Putting Eqs. (42) and (53) into Eq. (51) and solving for $P(r)$, we get

$$\tilde{P}(r) = \frac{\pi}{2} i \mu_0 c r_0 J_b^2 H_1^1(ik_0 r_0) J_1(ik_0 r) - iAkr N_b \gamma_0 m(\omega - kv_0) \quad (55)$$

Linearizing Eq. (26) with the aid of Eqs. (32), (54), and the vector equation $(\nabla \bar{v}) \cdot \bar{e}_r = \nabla v_r$, we get

$$(\omega - kv_0) \bar{n} = Ak \left(\frac{1}{r} \bar{e}_\theta + k \bar{e}_z \right) \quad (56)$$

Inserting this last result and Eq. (54) into the jump-condition, Eq. (16a'), a constraint on the frequency, ω , emerges. Performing the above substitutions, we find that

$$Ak \left(\frac{\omega}{\omega - kv_0} \right) = 0. \quad (57)$$

As $Ak \neq 0$, to satisfy Eq. (57), we must require that inequalities

$$|\omega| \ll |kv_0|, \quad (58)$$

and Eq. (25) serve as definitions of what we mean by low frequency.

The arbitrary constants c_0 and A appearing in Eqs. (34), (54), and (55) are related by the fact that there is a connection between J_b^* and \bar{v} , the perturbed beam velocity. Appealing to the fact that the beam must move with its surface, we get the desired relationship,

$$A = ic_0(\omega - kv_0)/k. \quad (59)$$

Finally, upon substituting Eqs. (55) and (59) into Eq. (17a'), we get the desired dispersion equation for hose instability,

$$(\omega - kv_0)^2 = \omega_B^2 [1 - i\pi J_1(ik_0 r_0) H_1^1(ik_0 r_0)], \quad (60)$$

where

$$\omega_B^2 = \frac{\mu_0 e^2 N_b v_0^2}{2\gamma_0 m} = \frac{e^2 N_b}{2\epsilon_0 \gamma_0 m} \left(\frac{v_0}{c} \right)^2.$$

VI ANALYSIS OF THE DISPERSION EQUATION

It is important to remember that Eq. (60) is valid only for low frequencies and for long wavelengths. The equation was obtained under the assumptions that the displacement current is negligible compared to the conduction current, Eq. (25), that the phase velocity of the wave is much less than the beam velocity which is highly relativistic, Eq. (58), and that the wavelength of the disturbance is much longer than the beam radius, $kr_0 \ll 1$.

As indicated earlier, in solving the dispersion equation we will specify the wave number, k , and assume that it is real. The wave frequency, ω , will be expressed as a function of k and the other parameters of the beam-plasma system. With the time development of the system of the form $\exp i\omega t$, instability is indicated by complex ω 's with negative imaginary parts. Physically, the instability is due to the drag exerted on the beam by the currents induced in the plasma, which oppose the motion of the lines of force of the pinch field. This drag is in the same direction as the mean transverse velocity of the beam electrons and results in an exponential growth of the oscillations.

In most plasmas, $|\omega_i/\omega_e| \ll 1$ and Eq. (9) can be approximated by

$$\sigma = \frac{\epsilon_0 \omega_e^2}{\nu + i\omega} = \frac{\epsilon_0 \omega_e^2}{\sqrt{\omega^2 + \nu^2}} \exp i\Theta \quad (61)$$

where

$$-\frac{\pi}{2} \leq \Theta = -\arctan \frac{\omega}{\nu} \leq 0. \quad (62)$$

The solution of Eq. (60) for arbitrary values of the parameters is only possible with the aid of a computer. We shall give a poor man's discussion of the solution by confining ourselves to the following special conditions: (a) a collisionless plasma, $\Theta = -\pi/2$; (b) the thin-beam case when the skin depth is much greater than the beam radius, $|k_0 r_0| \ll 1$; (c) the thick-beam case when the skin depth is much less than the beam radius, $|k_0 r_0| \gg 1$.

A. A Collisionless Plasma

In the complete absence of collisions in the plasma, $\nu = 0$, and $k_0 r_0 = \omega_e r_0 / c$ is a real positive quantity, independent of ω . For real z it is most convenient to express the Bessel and Hankel functions of purely imaginary argument in terms of the modified Bessel functions $I_1(z)$ and $K_1(z)$. According to McLachlan [11]

$$J_1(iz) = iI_1(z), \quad H_1^1(iz) = -\frac{2}{\pi} K_1(z), \quad \text{real } z, \quad (63)$$

and putting Eq. (63) into Eq. (60), the dispersion equation becomes

$$\omega = kv_0 \pm [1 - 2I_1(k_0 r_0)K_1(k_0 r_0)]^{\frac{1}{2}}. \quad (64)$$

When $k_0 r_0 \ll 1$, $I_1(k_0 r_0) \doteq k_0 r_0 / 2$, $K_1(k_0 r_0) \doteq 1 / (k_0 r_0)$,

and when $k_0 r_0 \gg 1$

$$I_1(k_0 r_0) \doteq \exp(k_0 r_0) / [2\pi k_0 r_0]^{\frac{1}{2}},$$

(65)

$$K_1(k_0 r_0) \doteq \exp(-k_0 r_0) / [\pi / (2k_0 r_0)]^{\frac{1}{2}}.$$

By consulting tables of $I_1(x)$ and $K_1(x)$, such as page 1925 of Morse and Feshbach [5], and pages 237-242 of Jahnke and Emde [6], it is readily verified that for real $k_0 r_0$, $1 - 2I_1(k_0 r_0)K_1(k_0 r_0) \geq 0$. Hence $\text{Im } \omega = 0$, and the conclusion is that in the absence of collisions the beam-plasma system is stable. As a collisionless plasma does not exist in nature, we do not attach much significance to this prediction.

B. A Thin Beam

In a thin beam,

$$|k_0 r_0| = \frac{\omega_e r_0}{c} \left| [\omega/(\omega - i\nu)]^{\frac{1}{2}} \right| \ll 1. \quad (66)$$

The solution of the dispersion equation in this approximation is still difficult and for that reason we will also assume that we are dealing with a collision-dominated plasma when $|\omega| \ll \nu$. Let

$$k_0 r_0 \doteq \frac{\omega_e r_0}{c} [\omega/\nu]^{\frac{1}{2}} \exp i\pi/4. \quad (67)$$

For small arguments [11]

$$J_1(ik_0 r_0) \doteq ik_0 r_0/2 \quad (68)$$

$$H_1^1(ik_0 r_0) \doteq -\frac{2}{\pi k_0 r_0} \left[1 + \frac{k_0^2 r_0^2}{2} \ln |(k_0 r_0)| \right],$$

and the dispersion equation becomes

$$(\omega - kv_0)^2 = -\frac{k_0^2 r_0^2 \omega_B^2}{2} \ln |(k_0 r_0)| = i\Omega\omega, \quad (69)$$

where

$$\Omega = - \frac{\omega_e^2 r_o^2 \omega_B^2}{2c^2 v} \ln |(k_o r_o)|. \quad (70)$$

With Eq. (67) holding, Ω depends logarithmically on ω and can be assumed to be a constant in the first approximation. Equation (69) can be written

$$(\omega - kv_o)^2 - i\Omega(\omega - kv_o) - i\Omega kv_o = 0, \quad (71)$$

or

$$\omega = kv_o + \frac{i}{2} [\Omega \pm (\Omega^2 - 4i\Omega kv_o)^{\frac{1}{2}}]. \quad (72)$$

When $kv_o \ll \Omega$, we may expand the radical in Eq. (72) by using the binomial theorem. Of the two roots, one is unstable, $\text{Im } \omega < 0$, and satisfies the low-frequency condition, Eq. (58), also. It is

$$\omega = \frac{2k^3 v_o^3}{\Omega^2} - \frac{ik^2 v_o^2}{\Omega}, \quad kv_o/\Omega \ll 1, \quad |\omega| \ll v. \quad (73)$$

For these waves, the e-folding time is $\Omega/(kv_o)^2$.

In the opposite limit when $kv_o \gg \Omega$, a similar expansion shows that there is an unstable root:

$$\omega = kv_o - \sqrt{(\Omega kv_o/2)} - i\sqrt{(\Omega kv_o/2)}, \quad kv_o/\Omega \gg 1. \quad (74)$$

As this root does not satisfy the condition $|\omega| \ll kv_o$, it is not an acceptable solution and is to be rejected.

C. A Thick Beam

In a thick beam,

$$k_o r_o = \frac{\omega_e r_o}{c} \sqrt{[\omega/(\omega - iv)]} \gg 1. \quad (75)$$

For such large arguments we may use the asymptotic expansions of the Bessel functions [11]:

$$J_1(ik_0 r_0) \doteq \frac{1}{\sqrt{2\pi k_0 r_0}} \exp k_0 r_0 \quad (76)$$

$$H_1(ik_0 r_0) \doteq \sqrt{\left[\frac{2}{\pi k_0 r_0}\right]} \exp -(k_0 r_0) .$$

Inserting Eq. (76) into Eq. (60), the dispersion equation becomes

$$(\omega - kv_0)^2 = \omega_B^2 \left[1 - \frac{1}{k_0 r_0}\right], \quad (77)$$

with $k_0 r_0$ given by Eq. (75). Equation (77) can be solved by the standard iterative procedure. In the first approximation, its solution is

$$\omega_{\pm}^{(1)} = kv_0 \pm \omega_B . \quad (78)$$

Putting $\omega_{\pm}^{(1)}$ into Eq. (75) we get an estimate of $(k_0 r_0)^{-1}$, which we shall call ϵ_{\pm} . In the next approximation,

$$\omega_{\pm}^{(2)} = kv_0 \pm \omega_B [1 - \epsilon_{\pm}/2] . \quad (79)$$

However, this iterative process is meaningless physically, because the macroscopic analysis which led to Eq. (60) tacitly assumes that the beam electrons are influenced by the quasi-static electromagnetic field during the course of one betatron oscillation, ω_B . In other words, the Doppler-shifted frequency, $\omega - kv_0$, satisfies the inequality,

$$|\omega - kv_0| \ll \omega_B . \quad (80)$$

As the roots of Eq. (77) do not satisfy this condition, we must reject them and attach no physical significance to them.

VII DISCUSSION

The hose instability of a finite, relativistic beam penetrating a plasma has been studied by other physicists using different but reasonable models of the beam-plasma system. Apparently, the dispersion equation is not very sensitive to the model used, for they obtained the identical dispersion equation, Eq. (60). The earliest of these known to the author is by G. Ascoli [9], who credits the heuristic and strongly physical derivation he gives, to C. Longmire, M. Rosenbluth, and N. Christofilos. Except for slight differences in notation, Ascoli's Eq. (37) on page 30 is identical to our Eq. (60). Additionally, he also gives a discussion of sausage instability, the $m = 0$ mode. Readers lacking Ascoli's strong physical intuition will find his paper somewhat hard to follow. It is hoped that they have found this paper more straightforward and understandable.

In a pellucid paper, S. Weinberg [8] has derived and analyzed the dispersion equation for hose instability of a finite beam ab initio. Although he used a slightly different model from the one employed here, he obtained the identical dispersion equation and arrived at the same conclusions. Weinberg's paper should be read in conjunction with a paper by Rosenbluth [10] which is rather difficult to understand because of the host of approximations made to omit terms which were supposed to be small. Whereas Rosenbluth attempted to treat the beam dynamics by solving the relativistic Vlasov Equation, Weinberg simply used Newton's Second Law of Motion. This simplifies the treatment greatly and makes the final conclusion much more plausible.

In a long and skillfully written technical report deserving of much careful study, Mjolsness [7] has studied the stability of a relativistic particle beam passing through a plasma. In Chapter II, he analyzes the hose instability in a much more detailed and convincing manner than Ascoli did, and the dispersion equation which he obtains, his Eq. (90), is identical with our Eq. (60). Besides using a model which is essentially the same as the one used here, he also derives the identical dispersion equation by following a technique employed by Rosenbluth in studying the same problem [10]. Defining \bar{a} as the local surface displacement vector at the beam surface, he obtains the dispersion equation by integrating the \bar{a} component of the momentum equation over a beam cross section. By invoking a weak assumption on the pressure tensor, his Eq. (108) [recall that we assumed that the pressure was a scalar, Eqs. (31) and (55)], he obtained the dispersion equation.

In an accurate analysis of this problem, the beam and the plasma should both be treated microscopically. The mathematical difficulties encountered in such an ambitious program are insuperable. As a compromise, Mjolsness has given a microscopic analysis of the beam and treated the plasma macroscopically. Specifically, he has solved the collisionless Boltzmann equation for the beam and treated the plasma as an ohmic medium with a tensor conductivity. The vertiginous details are given in Chapters 4 and 5 of Mjolsness's report and interested readers are referred to it for details.

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GLOSSARY

| | |
|-------------|--|
| A | an arbitrary constant, Eqs. (53) and (59). $[A] = \ell^2 t^{-1}$ |
| b | used as a subscript, b means beam |
| \bar{B} | magnetic induction |
| c | speed of light in a vacuum, 3×10^8 m/sec |
| c_0 | a constant, Eq. (34). c_0 is a length. |
| e | magnitude of the electron charge |
| \bar{e}_r | unit vector in the r direction |
| \bar{e}_z | unit vector in the z direction |
| \bar{E} | electric field |
| h | Debye length of plasma electrons, $h = [\epsilon_0 K T_p / (n_0 e^2)]^{1/2}$ |
| i | imaginary unit, $i^2 = -1$ |
| \bar{J} | current |
| \bar{J}^* | surface current at the beam-plasma interface, Eq. (22a) |
| \bar{k} | propagation vector, $k = 2\pi/\lambda$, k is a real quantity |
| k_0 | $k_0 = e^{i\pi/4} [\mu_0 \omega \sigma]^{1/2} = e^{i\pi/4} [\text{skin-depth}]^{-1}$ |
| K | Boltzmann's constant |
| m | electron rest mass, mode number in a normal mode analysis, Eq. (2) |
| M | ion rest mass |
| \bar{n} | outward directed, unit normal at the beam surface |
| n_0 | equilibrium electron number density of the plasma |
| N_0 | equilibrium ion number density of the plasma |
| N_{b0} | equilibrium electron density of the beam |
| p | used as a subscript, p means plasma |
| \bar{p} | $\gamma m \bar{v}$, linear momentum of beam electrons |

- P scalar pressure of the beam
- $q_1(r, \theta, z, t)$ perturbed value of the quantity $q(r, \theta, z, t)$, Eq. (2)
- r_0 equilibrium radius of the beam
- t time
- T temperature
- \bar{v} beam electron velocity, $\bar{v} = v_0 + v_1$, $|\bar{v}_1/\bar{v}_0| \lll 1$, $\bar{v}_0 = v_0 \bar{e}_z$
- γ $[1 - v^2/c^2]^{-\frac{1}{2}}$, the Lorentz factor, $\gamma_0 = [1 - v_0^2/c^2]^{-\frac{1}{2}}$, ratio of specific heat at constant pressure to that at constant volume, Eq. (3)
- $\delta(r)$ Dirac delta function, $\int f(r)\delta(r)dr = f(0)$
- ϵ positive, infinitesimal
- ϵ_0 permittivity of free space, $\bar{D} = \epsilon_0 \bar{E}$
- θ azimuthal angle in cylindrical coordinates, $x = r \cos \theta$,
 $y = r \sin \theta$
- Θ phase of the plasma conductivity, Eq. (13)
- λ wavelength
- μ_0 permeability of free space, $\bar{B} = \mu_0 \bar{H}$
- ν electron-ion collision frequency of the plasma, Eq. (14)
- ρ mass density of beam electrons, $\rho = N_b m$
- σ scalar conductivity of the plasma, Eq. (9)
- $\bar{\psi}$ $\bar{\psi} = \bar{\psi}(r)$, radial dependence of the electric vector, Eq. (36)
- ω wave frequency, as we assume time dependence $\exp i\omega t$,
Im $\omega < 0$ indicates instability
- ω_B betatron frequency, $\omega_B^2 = \frac{\mu_0 e^2 N_b v_0^2}{2\gamma_0 m} = \frac{N_b e^2}{2\epsilon_0 \gamma_0 m} \frac{v_0^2}{c^2}$
- ω_e $\omega_e^2 = \frac{n_0 e}{\epsilon_0 \mu}$, Eq. (8)

$$\alpha = \frac{N_o(m+M)}{n_o m + N_o M} \doteq 1, \text{ Eq. (8)}$$

$$\Omega = - \frac{\omega_e^2 r_o^2 \omega_B^2}{2c^2 v} \ln |k_o r_o|, \text{ Eq. (70)}$$

$J_1(z)$ Bessel function of order unity

$H_1^1(z)$ Hankel function of the first kind or order unity

$I_1(z)$ modified Bessel function of order unity, $J_1(iz) = iI_1(z)$,
z real

$K_1(z)$ modified Bessel function of order unity, $H_1(iz) = -\frac{2}{\pi} K_1(z)$,
z real

$\tilde{P}(r)$ radial factor of the pressure,
 $P(r, z, \theta) = \tilde{P}(r) \exp i(\omega t - kz - \theta)$

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